Tolman-Ehrenfest Effect in Reissner-Nordstrom Geometries: *Charge, Gravity and Temperature*

by

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Abstract

In General Relativity, the naive notions of equilibrium temperature are replaced with temperature gradients and well-understood proofs have demonstrated the relationship between gravity and equilibrium temperatures. The 2018 Gravity Research Foundation's first prize winning essay establishes that only gravitational fields can create equilibrium temperature gradients using the clever addition of Electric Fields in a Maxwellian argument.

In my work I try to answer the question, whether equilibrium temperature gradients 'feel' the effects of electric field. To do so, I first revisit the Maxwellian argument of two columns along with the aforementioned analysis of the Tolman gradient. Then, I argue that though Electric Fields do not cause temperature gradients, their presence is not inconsequential. I do so by studying a body in thermal equilibrium in a Resissner-Nordstrom geometry and why it differs from the temperature gradients in a simple Schwarzchild geometry.

Contents

1. Thermodynamics and Relativity

1.1 Maxwell's Two Column Argument

Let us start by revisiting an argument by Maxwell that is more than a century old (Maxwell 1868). The original purpose of the argument by Maxwell was to demonstrate that temperature gradients cannot exist in bodies at thermal equilibrium.

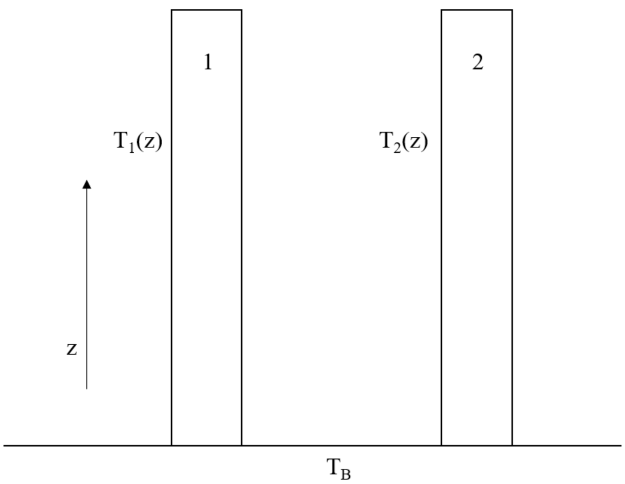

Figure 1.1: The Hypothetical Two Column Setting

Imagine two columns of arbitrary materials standing on a conducting surface. The base of the two columns are in thermal equilibrium with the conducting surface that they stand on as displayed in Fig. 1.1. Let us start by assuming that, though the columns are in thermal equilibrium, they demonstrate a temperature gradient across the z-axis.

If at any given height z we come across a situation such that $T_1(z) > T_2(z)$ (or vice versa, since our system is completely arbitrary) then we can connect a conducting rod at that height. On this rod, heat flows from 1 towards 2. This added heat in 2 would flow towards the base $T_2(0)$, which heats up the base of 1 as well via the conducting surface over which the columns stand. When the base of column 1 gets heated, the heat flows upward in the column again forming a cycle (see Fig. 1.2).

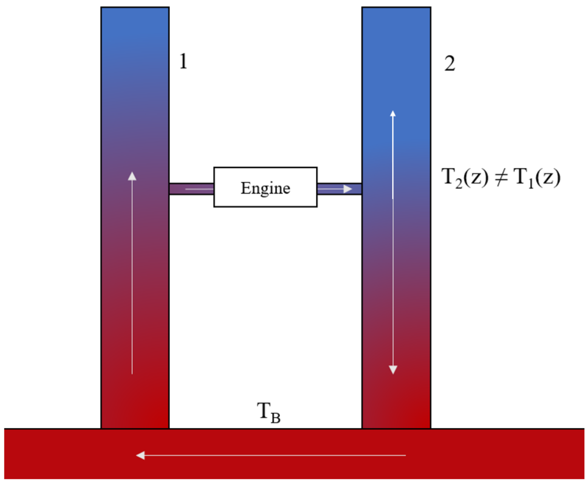

Figure 1.2: Inequality in Temperature Gradient Creating Perpetual Motion Machine

We can place an engine on the conducting rod at height z. This renders us with a perpetual motion machine with the two columns perpetually acting as sources and sinks at different temperatures even at thermal equilibrium. This is an absurdity and clearly violates the second law of thermodynamics. Therefore, our assumption that $T_1(z) \neq T_2(z)$ must be false.

From here, Maxwell argues that since we do not see a temperature gradient in a column of ideal gas, no substance can demonstrate a temperature gradient at equilibrium. If such a gradient exists at all, it must be universal. Otherwise, it allows for the violation of Clausius's version of the second Law.

1.2 Photon Gas Columns and Gravity

Now, we fast forward in time and acquire some of the important insights provided to modern physics by the Theory of Relativity. Using Energy-Momentum Tensor of a perfect fluid, Tolman and Ehrenfest did important calculations to demonstrate that presence of a static gravitational field causes a temperature gradient even at thermal equilibrium (Tolman and Ehrenfest 1930). This temperature gradient could be expressed by -

$$T(z) = \frac{T_0}{\sqrt{-g_{tt}(z)}}$$

Here, $T(z)$ is the temperature at thermal equilibrium that varies across position, g_{tt} is the time-like component of the metric tensor for any given static field and T_0 is a constant. Thus, at thermal equilibrium, it is T_0 that remains constant spatially and temporally and not the locally measured temperature. We call this *Tolman Temperature* or *Ehrenfest Temperature*. There are some subtle difference in the notations used by Tolman and Ehrenfest in their original paper and the notation that I employ.

Later, these results were generalized for any stationary gravitational field (Santiago and

Viser 2018). As long as the fluid follows integral curves of a time-like Killing Vector K, the temperature gradient is given by

$$T(z) = \frac{T_0}{\|K\|}$$

Like the previous case, T_0 is truly the temperature that remains constant at thermal equilibrium across space and time. Whereas, the temperatures measured locally $T(z)$ would display a gradient depending on the norm of the time-like Killing Vector. For the purposes of this report, we shall only concern ourselves with static spacetimes (Schwarzchild Spacetime and Reissner-Nordstorm Spacetime) and would not be requiring the stationary solutions.

Temperature Gradient Derivation for Static Spacetime

I will present a modified version of a simple proof of the temperature gradient expression using a photon gas column as discussed in Santiago and Viser 2019. This proof does not require us to delve deep into concepts from General relativity and only borrows the idea of gravitational redshift form the theory. However, unlike Santiago and Viser 2019, who used a constant gravitational field, I would employ a spherically symmetric gravitational field. Let us go through the proof now.

Consider a spherically symmetric gravitational field around a massive body and a long photon gas column placed offset from radial direction as demonstrated in Fig. 1.3. Suppose an observer situated a large distance away from the photon gas column concludes that the spectral radiance of each section of the column demonstrate a peak - invariant of time - at the same wavelength. From Wein's Displacement Law, the observer infers that the body is in thermal equilibrium because the temperature of the column is spatially and temporally constant.

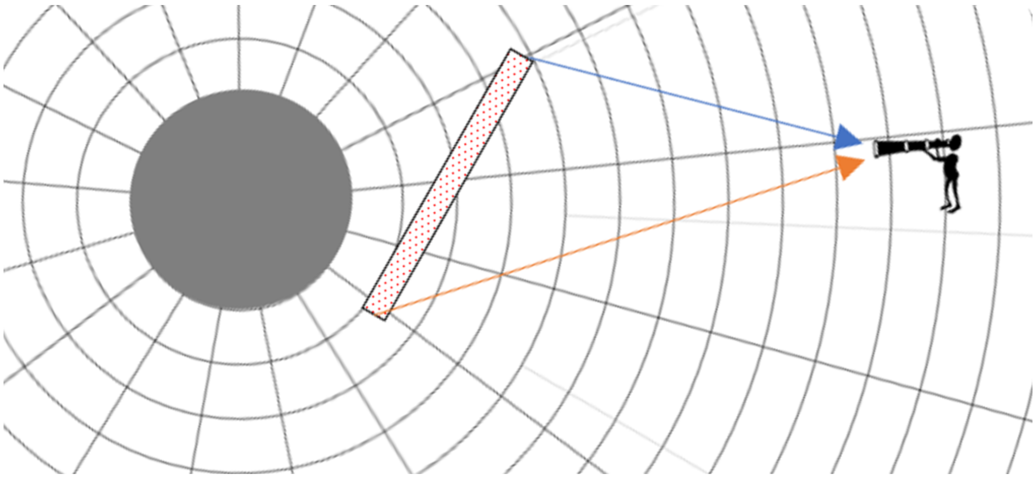

Figure 1.3: An observer observing the photons leaking from a photon gas column near a massive body

However, we know in General Relativity, a photon's motion through a gravitational well leads to energy-loss and, subsequently, red-shift in wavelength. The expression to evaluate gravitational red-shift in spherically symmetric Schwarzchild metric is well understood to be -

$$\frac{\lambda_\infty}{\lambda_e} = \left(1 - \frac{R_s}{r}\right)^{-1/2}$$

where λ_e is the wavelength at emissions, λ_∞ is the wavelength observed at infinity, $R_s = 2GM/c^2$ is the Schwarzschild radius of the massive body and r is the radius at which the photon was initially emitted. For our observer situated large distance away from the body, $\lambda_o = \lambda_\infty$ is given by -

$$\lambda_0 = \lambda_e \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}$$

From Wein's Displacement Law, we know $\lambda_{o_{max}} T_o = \lambda_{e_{max}} T_e$. Therefore, the temperature recorded by the observer would be off by a factor of -

$$T_e = \frac{T_o}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

In our hypothetical scenario, however, different sections of the photon column are clearly at different distances from the center of the massive body. This means one end of the column should experience a larger redshift than the other and this redshift should be reflected in the Blackbody curve recorded by the observer. Thus, instead of seeing a constant spectral distribution throughout the column, the observer should be seeing displaced intensity peaks at different positions of the column. This is a clear contradiction. The only way the observer would observe spatially constant temperature throughout the column is if the there existed a temperature gradient that would exactly cancel out the gravitational redshift experienced by the photons. Therefore, we end up with an expression for locally measured temperature of the photon gas column that - though temporally constant - is a function of position -

$$T(r) = \frac{T_o}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

We were working in the Schwarzschild Geometry and in our case the metric is defined by -

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) c^2 dt^2 + (1 - \frac{R_s}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

On comparing with our expression for locally measured temperature, it becomes clear that the Temperature gradient of the photon gas column follows the expression -

$$T(r) = \frac{T_0}{\sqrt{-g_{tt}(r)}} \tag{1.1}$$

where T_0 is constant as described earlier. Notice that the gradient is independent of time. This is a counter-intuitive result because it implies that though there exists a temperature gradient in the photon gas column, the heat does not flow from hotter region to colder region. As we shall see later, in relativistic thermodynamics (as in other aspects of relativity), there is a need to differentiate between 'coordinate' temperature and 'proper' temperature [1].

1.3 Connecting the Two Pieces

In Section 1.1 we established using a Maxwellian argument that if there exist two unequal temperature gradients, we can run a perpetual motion engine. In Section 1.2, we established that a photon gas column must have a temperature gradient for it to produce consistent results with observations. If we connect these two pieces of the argument, we can arrive at a general result about equilibrium temperature gradients i.e. Eq. 1.1 models the temperature gradient for all objects in static spacetimes.

It is simple to prove this using an analogous argument from Section 1.1. Let us start by assuming that only the photon gas column demonstrates a gradient at thermal equilibrium and another column made of some other material (say, ideal gas) does not. If we keep them parallel and very close to each other (Fig. 1.4) and connect them via conducting surface at the base then we arrive at the exact same scenario mentioned in Fig. 1.2. Therefore,

[1] This is not standard terminology and has been introduced only for convenience

on connecting a conducting rod between the two columns, we can extract work from heat transfer indefinitely. Although, in the non-relativistic limit $c \to \infty$ the gradient would be indeed in the limit $\nabla T(r) \to 0$.

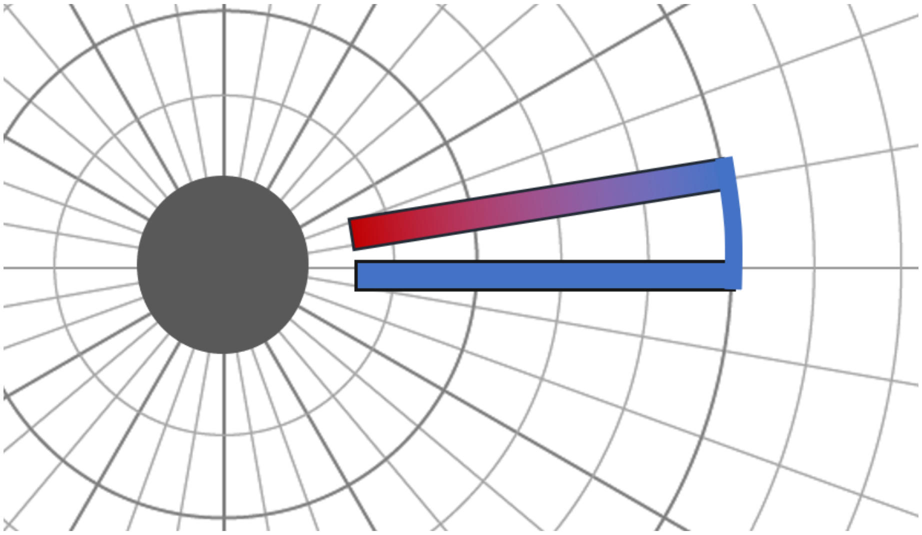

Figure 1.4: An observer observing the photons leaking from a photon gas column near a massive body

To avoid the problem posed by perpetual motion, we must accept that all temperature gradients at thermal equilibrium in a certain geometry must follow the same relation described by Eq. 1.1. The temperature gradient is caused due to spacetime curvature around the massive body and, therefore, all massive particles and radiation interacting with the spacetime curvature should experience the same temperature gradient (to eliminate the possibility of perpetual motion). This establishes the *Universality of Gravity*.

2. Geometry and Temperature

2.1 Temperature Gradients due to $\vec{E}$ (or Lack Thereof)

Having established the universality of gravity and the temperature gradient associated with gravity at thermal equilibrium, we move our attention to the possibility of temperature gradients formed by electromagnetic field. The following is also a part of Santiago and Viser 2019 and is developed over Maxwell's two column argument. We consider a similar apparatus as described earlier with few minor adjustments. Suppose one of the columns is filled with very low density electron gas and the entire apparatus is subjected to an Electric Field $\vec{E}$ as in Fig. 2.1. Does $\vec{E}$ produce a temperature gradient at thermal equilibrium? Let's start by assuming it does. What happens next?

If there is a temperature gradient produced due to the electric field then it must only affect those particles that interact with $\vec{E}$ to have any causal relationship in the first place. If the adjacent column is made of non-interacting particles (such as Neutron Gas) then $\vec{E}$ has no causal influence over the second column. Thus, we are in a situation where there exists a temperature gradient in the electron gas column and no temperature gradient in the neutron gas column at thermal equilibrium. After having repeated the Maxwell's argument twice,

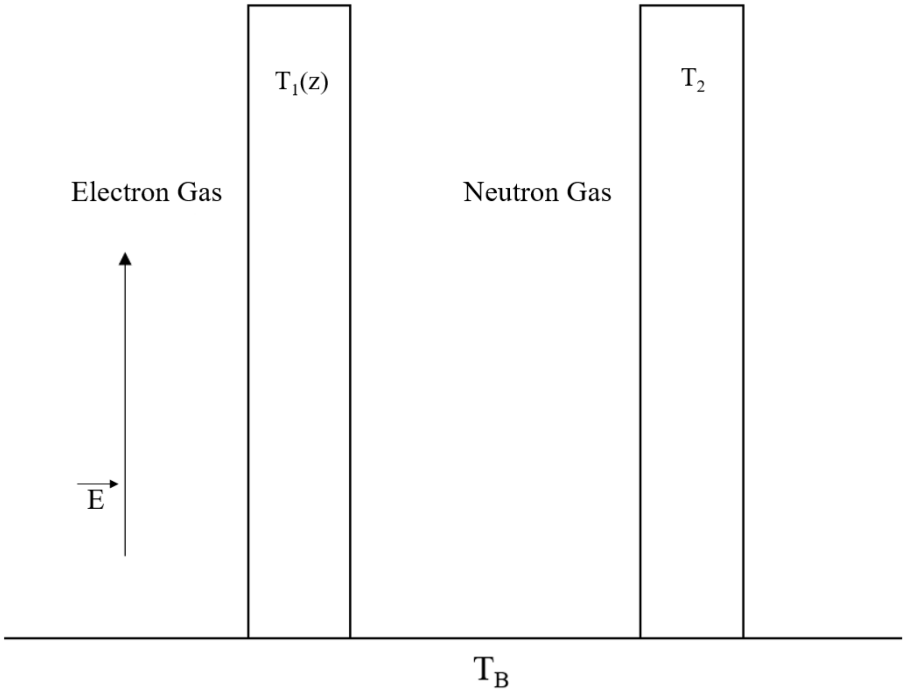

Figure 2.1: Electron Gas and Neutron Gas Columns Exposed to $\vec{E}$

this should be immediately alarming. At the risk of being redundant, let me explicitly state the problem again. If at any height in the system at thermal equilibrium, the temperature of the two columns is in in equal, we can run an engine between them using the thermal gradient *ad infinitum*.

This result can be further generalized. Any force for which there exists particles that do not interact with the force cannot produce a temperature gradient at thermal equilibrium. Gravity is universal and, hence, causes Tolman-Ehrenfest Effect at thermal equilibrium. Electric force is not universal and, hence, thermal equilibrium is unaffected by it.

2.2 Reissner–Nordström Geometry

Notice that we ignored gravity in the previous section. Let us factor effects of gravity again. Does presence of electric field affect the temperature gradient at equilibrium. The answer to this is slightly nuanced. As we have already established, electric force should have no contribution in the development temperature gradient. For now, we take a brief detour to cover two of (classical) Maxwell's equations -

$$\nabla.\vec{E} = \frac{\rho}{\varepsilon_0} \text{ and } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Clearly, to have an electric field, one either needs a charge density or a changing Magnetic Field. In other words, electric field lines either originate/end on charges or form closed loops.

In general relativity, anything with Energy-Momentum affects the geometry of space-time. The simplest analysis of gravitational influence of massive charged objects is possible **Reissner–Nordström** metric, which is a static spacetime. The metric that describes the spacetime near a charged spherically symmetric Blackhole [1] is given by -

$$ds^2 = -\Delta c^2 dt^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

[1] If the charged blackhole is also spinning then we call it a Kerr-Newman geometry. Since we are not looking at stationary spacetimes and magnetic mono-poles do not exist, we will not worry much about the magnetic component for now.

where the coefficient Δ is given by -

$$\Delta = \left(1 - \frac{R_s}{r} + \frac{R_Q^2}{r^2}\right)$$

Like earlier, $R_s = 2GM/c^2$ is the Schwarzschild radius and $R_Q = [Q^2 G]/[4\pi\varepsilon_0 c^4]$ is a characteristic length defined by the net charge content of the body. Clearly, we set $Q = 0$, we simply get a Schwarzschild geometry. For emphasis, I reiterate the implication, presence of charge affects the geometry of spacetime.

It is instructive to think of the massive bodies as blackholes because of the exaggerated gravitational effect they cause. Otherwise, the argument is completely general and works around any charged massive body with spherical symmetry.

2.3 Do Temperature Gradients feel $\vec{E}$?

We now return back to the original question - are temperature gradients at equilibrium (in static spacetime) affected by the presence of electric field? If the electric field could influence the temperature gradient at thermal equilibrium then it would allow for the existence of perpetual motion machines. Therefore, we rule out the possibility of a contribution by the electric field at thermal equilibrium. However, in a static spacetime, electric fields cannot exist without a charge. Since the existence of charge affects the spacetime metric and the spacetime metric affects the temperature gradient, the presence of $\vec{E}$ does indeed contribute to the temperature gradient in an indirect manner.

Obviously, in the our routine thermodynamic experiments, ∇T is incredibly small (with or without the presence of charge). For all practical purposes, we ignore the Tolman gradient but it must exist to ensure the logical consistency of our theories.

3. Final Remarks

The work I presented was an elementary clarification regarding the nature of causal relationship between thermal equilibrium temperature gradients and presence of electric field. To build towards the final conclusion, I introduced Maxwell's two column argument and associated it with a slightly modified version of the claims presented by Santiago and Viser 2019. Then, on studying the presence of electric field carefully and keeping universality of gravity in mind, I established that electric fields influence temperature gradients even though electromagnetism is not universal like gravity. Since *this influence is carried out via gravity itself,* the conclusion of Santiago and Viser 2019 remains consistent with my result - Gravity is the only force capable of creating temperature gradients at equilibrium.

Even though the results of these thought experiments may seem inconsequential due to their negligible scales, the existence of these temperature gradients at thermal equilibrium requires a reformulation of other laws of thermodynamics. The definition of Temperature given by Zeroth Law is not consistent with GR anymore. The directional of heat flow prescribed by Second Law is not persevered due to the existence of stable gradients. Fortunately, the field of relativistic thermodynamics is well established and such issues have already been resolved.

Bibliography

Maxwell, James Clerk (1868). “On the dynamical Theory of Gas”. In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 35. DOI: https://doi.org/10.1098/rstl.1867.0004.

Santiago, Jessica and Matt Viser (2018). “Tolman-like temperature gradients in stationary spacetimes”. In: *Physical Review D* 98. DOI: https://doi.org/10.1103/PhysRevD.98.064001.

— (2019). “Tolman temperature gradients in a gravitational field”. In: *European Journal of Physics* 40. DOI: https://doi.org/10.1088/1361-6404/aaff1c.

Tolman, Richard C and Paul Ehrenfest (1930). “Temperature Equilibrium in Static Gravitational Field”. In: *Physical Review* 36. DOI: https://doi.org/10.1103/PhysRev.36.1791.