Schwarzschild Black-Hole Behavior in a CMBR Bath Evolving under a Flat FRW Cosmology:

An Thermodynamic Exploration

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Abstract

Black-holes are immensely useful areas of interest in general relativity. Previous works of Bekenstein, Zeldovich, Penrose, Hawking and others have helped formulate an understanding of Black-Holes treating them as thermodynamic objects. Employing general relativistic results and introducing quantum fluctuations yield, via sophisticated derivations, relatively elementary expressions resembling classical thermodynamic results. One such important result comes in the form of 'Hawking Radiation', which claims Black-Holes emit radiation and possess a surface temperature. Another concept ubiquitous to discussions in cosmology is the Cosmic Microwave Background radiation that permeates all of observable universe. Black-holes, therefore, both *emit* Hawking Radiation and *absorb* Cosmic Microwave Background radiation.

Additionally, in our expanding universe, the temperature of CMB evolves over time as a function of the scale factor due to redshift. Thus, evolution of Black-Hole surface temperature, when soaked in a bath of changing CMB radiation, towards an equilibrium temperature becomes an interesting problem. Do Black-holes evaporate? Which thermodynamic contribution dominates such dynamics? These are the kinds of question I shall explore in this paper after introducing some basic ideas related to Blackholes and CMBR.

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1. **Introduction**

The primary objective of this project is to employ a law of blackhole (BH) thermodynamics with the consideration for the presence of dynamic Cosmic Microwave Background Radiation (CMBR) and gain insights about BH time evolution. To keep the report as complete, comprehensive and self-contained as possible, I have structured it in a way such that first elementary ideas about CMBR and BHs (within the context of thermodynamics) are introduced before jumping into the core content.

2. **Cosmic Microwave Background Radiation**

Though the night sky appears dark to human eyes, to a sensitive enough telescope there appears to be a faint glow spread across the sky primarily in the microwave region. The accidental discovery of this microwave background radiation resulted in a Nobel prize for Wilson and Penzias in 1978 because it served as an important evidence for the standard Big Bang cosmological model. The principle theme of this section is to find an expression for temperature of CMBR in terms of time.

2.1 *Early Universe and Recombination*

According to the Big Bang inflationary model, our universe during its infancy was much denser, hotter and filled with an opaque hydrogen plasma. However, as the universe expanded, it grew cooler and eventually reached a temperature that permitted the formation of neutral hydrogen atoms from the electrons and protons available (below 3000K, as modelled to first approximation by the *Saha* Equation). Unlike the free charges, neutral hydrogen atoms could not partake in Thomson scattering and the universe turned transparent. This period is named as the epoch of *Recombination*, which is a slight misnomer because the charges combined for the first time during this epoch.

2.2 *A Perfect Blackbody*

We have made very precise measurements of the CMBR and concluded that it behaves extremely close to how a perfect blackbody at a uniform temperature of $2.72548 \pm 0.00057K$ (refer Fig. 1) would behave. However, a raw observation of CMBR would not reflect this isotropy. There are several anisotropies in the CMBR as observed from Earth and the primary one is caused due to the Doppler effect of Earth's motion around the sun. Further anisotropies are caused due to the redshift of light as it moves through a gravitational well (Integrated Sach-Wolfe Effect), distortion due to high-energy electrons in galaxy cluster (Sunyaev-Zeldovich effect), density perturbations in early universe that correspond to large scale structures observed today and more. Characterizing and modelling the fluctuations in CMBR is a pursuit that involves rich physics (beyond the mandate of this paper) and is very valuable in establishing the robustness of Standard Model of Cosmology.

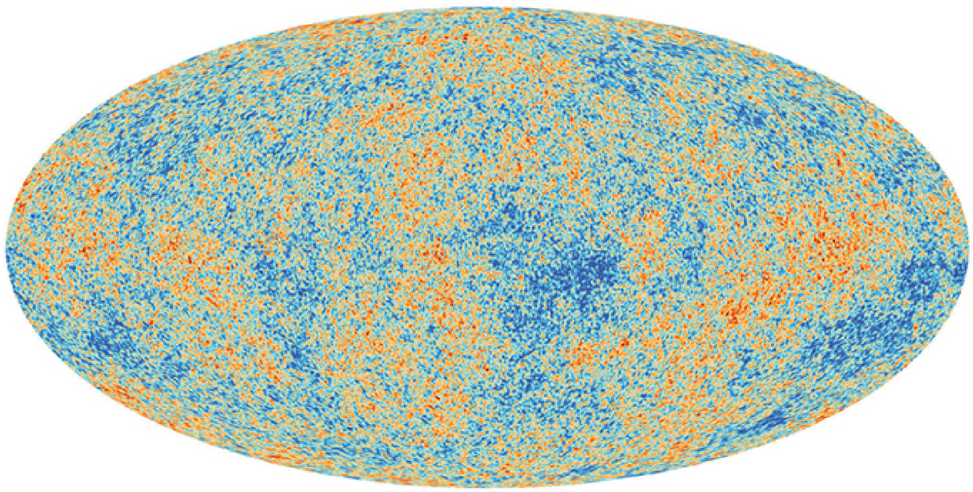

Figure 1: Latest probe of CMBR conducted by ESA's Planck satellite in 2013 demonstrating the difference between hot and cold regions of the magnitude of 0.00001 K after known anisotropies have been filtered out

As demonstrated by Fig. 2, CMBR is one of the most perfect physical examples of a Black Body that we have studied. Note that though CMBR has cosmological origins and is a (very) large scale phenomena, it is still essentially not much different from the cavity radiation analysed during the undergraduate thermal physics class.

2.3 *Expansion and CMB Temperature*

The temperature of CMB does not remain constant over time and is affected by Cosmological Redshift. To build towards a final expression, we first need to familiarize ourselves with the concept of cosmological red-shift. As per observations and theory, we live in an expanding universe. As the space between the observer (in our case Blackhole) and CMBR expands, the radiation gets red-shifted. This red-shift is the reason why the background radiation is currently in the microwave regime, even though it was much hotter when it was emitted during the recombination epoch. This

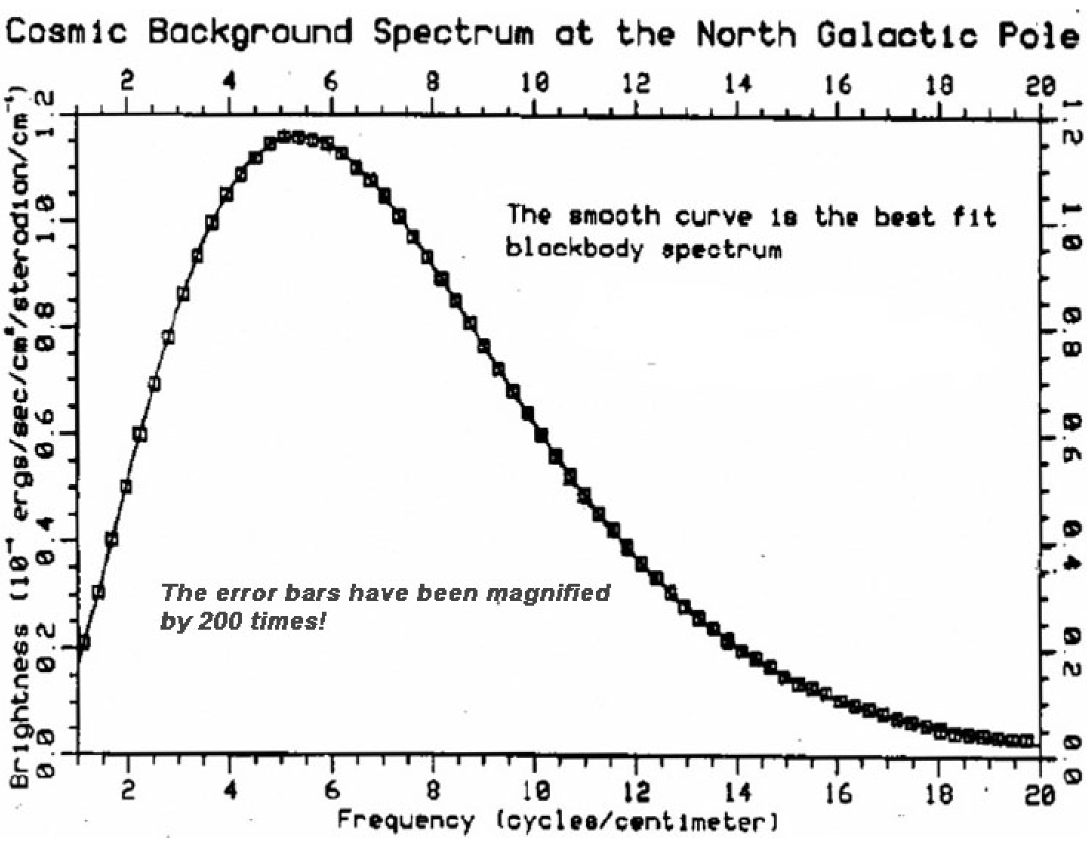

Figure 2: Data from the observations of Comsmic Microwave Background Radiation Spectrum tightly fitting a perfect Black Body Curve. Image Source: Duke University

cosmological red-shift is characterized by the red-shift parameter (generally denoted by z). We define the red-shift parameter as -

$$\lambda_o = \lambda_e(1 + z)$$

where λ_o is the observed red-shifted wavelength and λ_e is the wavelength during when emitted from the original source. This is a completely general formula and is not restricted only to the study of CMBR. Further, from Wein's Displacement Law, we know that for any blackbody -

$$\lambda_{max}.T = W$$

W being the Wein's constant. Using simple manipulations, we can get -

$$\lambda_o T_o = \lambda_e T_e \implies (1 + z)\lambda_e T_o = \lambda_e T_e$$

$$\implies T_o = \frac{T_e}{(1 + z)}$$

Therefore, the temperature of CMBR is related to the cosmological red-shift parameter. Remember, that in our expanding universe $z = z(t)$. Since the red-shift is caused due to the expansion of the universe, we bring into discussion another important variable in the study of cosmology - (nondimensionalized) scale factor $a(t)$. A widely understood and useful relation in cosmology (Carroll and Ostlie) is -

$$1 + z(t) = \frac{a(t)}{a(t_e)}$$

Where $a(t_e)$ is the scale factor of the universe at the time of emission (in simple words, numerator is scale factor 'now' and the denominator was the scale factor 'then'). Therefore, we end up with

$$T(t) = T_e \frac{a(t_e)}{a(t)} \tag{1}$$

Now, to understand time evolution of CMBR temperature, we need to understand time evolution of scale factor. This is far from elementary and it goes without saying that I would be using some results without demonstrating rigorous proofs of them. To make a crude analogy, we can think that the following analysis treats the universe as a fluid of sorts with galaxies acting as molecules in a gas. Though the analogy is incomplete, it is instructive to employ it when discussing density, pressure, expansion, contraction, continuity etc. of the universe. The expansion of the universe, at least within our FRW cosmology, can be modelled by the using any one of the two Friedmann Equations (2) (Hartle)

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) \tag{2}$$

and the fluid Equation (3).

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \tag{3}$$

Here, ρ is the matter-energy density in the universe, p is the pressure and k represents the curvature. It is also useful to define the Hubble Parameter as $H = \dot{a}/a$. We are restricting ourselves to a flat universe where $k = 0$, which is a reasonable consideration at the length scales we are concerned with.

Since the system of the Friedmann equations and fluid equation is not independent, one can use any two equations to derive the third. Remember ρ and p also change with the scale factor. This means we currently have three unknowns variables (a, p and ρ) and only two equations. Thus, we will need an *Equation of State* that associates pressure with density and is defined for a perfect fluid as -

$$p = w\rho$$

Further, the matter-energy density has three different contributions and each of them have a corresponding value of constant w [1]. Therefore, it is helpful to define density in terms of relative density factors of each contribution (Carroll and Ostlie). We also know the values of several of

[1] It is yet another simplification to assume w remains constant in time. When studying the behavior of w parameter in cosmology, we start approaching some open theoretical questions

the mentioned parameters at current time t_0 via the observational evidence collected by WMAP [2]. Note that ρ_c is the critical density and is defined as $3H^2/8\pi G$.

1. **Matter Contribution** This contribution is made by all the Baryonic and dark matter in the universe; $\Omega_{m,0} = \rho_m(t_0)/\rho_c = 0.27 \pm 0.04$. Of this, ordinary Baryonic matter makes up for only 17% and the rest is dark matter. On the length scales involved, the pressure contribution of matter is negligible ($w = 0$) and its density drops along with volume of the universe. So, $p_m = 0$ and $\rho_m(t) = \rho_m(t_0)/a(t)^3$.

2. **Radiation Contribution** This contribution is made by all the relativistic particles like photons and neutrinos; $\Omega_{r,0} = \rho_r(t_0)/\rho_c = 8.24 * 10^{-5} \pm 0.04$. As per special relativity, for radiation $p_r = (1/3)\rho_r$. Thus, w for radiation is (1/3). Radiation density also falls off with the volume of the universe but the radiation also loses energy due to expansion. So, the inverse proportionality is stronger giving us $\rho_r(t) = \rho_r(t_0)/a(t)^4$.

3. **Vacuum Energy Contribution** This contribution is made by Vacuum Energy (or so called Dark Energy) and is currently the dominant influence on dynamics at cosmological scales; $\Omega_{v,0} = \rho_v(t_0)/\rho_c = 0.73 \pm 0.04$. For the equation of state parameter w of vacuum energy, we do not have well accepted theories. We only know it has a negative value and it seems to be in the vicinity of -1. And from fluid equation, we get $\rho_v(t) = (1 + 3w)\rho_v(t_0)/a(t)^{3w+2}$.

On combining the equation of state, fluid equation and Friedmann acceleration equation and then substituting the density fractions and Hubble Parameter we get -

$$\ddot{a} = -\frac{4\pi G}{3}\rho_c \left[\frac{\Omega_{m,0}}{a^2} + \frac{\Omega_{r,0}}{a^3} + (1+3w)\frac{\Omega_{v,0}}{a^{3w+2}}\right] = \frac{-\dot{a}^2}{2a^2}\left[\frac{\Omega_{m,0}}{a^2} + \frac{\Omega_{r,0}}{a^3} + (1+3w)\frac{\Omega_{v,0}}{a^{3w+2}}\right]$$

This second order ODE can be solved numerically using the initial conditions $a(t_0) = 1$ and $\dot{a}(t_0) = H_0 = 7.1 * 10^{-11} yr^{-1}$

3. Black Hole Basics

Now that we have a brief understanding of the temperature of CMBR, let us equip ourselves with concepts from Blackhole thermodynamics that we need to model our system. As mentioned earlier, complete derivations of some of the equations are challenging endeavours that involve both General Relativity and Quantum mechanics. Instead of trying to develop a complete rigorous understanding

[2] There are scientists who raise contentions when the interpretations of WMAP results are discussed. Notably Prof. Subir Sarkar's group raises issues with model based inferences drawn from WMAP data. Upcoming telescope ELT would enable realtime cosmology and should be able to clarify some of the confusions regarding the nature of accelerated expansion of universe

of these, we shall have an application based approach towards the laws. Like before, the primary goal here is to reach an equation of temperature of black-hole as parameterized by time.

3.1 *Characterizing Schwarzschild Blackholes*

The simplest form of Blackhole that we can study is the Schwarzschild Black Hole. It possesses complete spherical symmetry, no charge and no angular momentum. On the other hand, the most general black hole that one might expect to encounter is the Kerr-Newman Blackhole which is characterized by only three variables - Mass, Charge and Angular Momentum. The idea that a black-hole can be described in complete detail by very few macroscopic properties should already start hinting towards the fundamental ideas of thermodynamics (Steane). Work can be extracted from a rotating blackhole and a charged blackhole. Since the analysis of work could be thermodynamic interest in some cases, I decided to mention it here. However, for our study, we are only concerned with the exchange of energy in our system. So, we shall restrict our mathematical formulation to that of a Schwarzschild BH but it is not too difficult to generalize it for any blackhole.

A Schwarzschild BH has an event horizon radius that is called the Schwarzchild Radius (parameterized by the mass) and it is calculated using the following expression.

$$R_s(M) = \frac{2GM}{c^2}$$

Naturally, the surface area of the Blackhole event horizon would then be

$$A_s(M) = 4\pi R_s^2 = \frac{16\pi G^2}{c^4} M^2 \tag{4}$$

Now, we move to the next important piece of the problem.

3.2 *Hawking Radiation*

Hawking predicted that Black-Holes must emit radiation due to quantum effects near the event horizon. This was a surprising idea because conventional BH wisdom prescribed that they suck everything in and do not radiate outwards. The colloquial explanation for this contradictory phenomena is that, due to quantum effects, pairs of virtual particles spontaneously come into existence in space. If this happens very close to the horizon, one of the particles gets sucked inside while the other escapes to infinity.

The existence of Hawking Radiation meant that there must be a temperature associated with the Black-Hole too, which now we simply term Hawking Temperature or temperature of the BH (Barry

P.). This temperature can be calculated using the following relation -

$$T_{BH} = \frac{\hbar K}{2\pi c k_B}$$

where, K is the surface gravity of the BH evaluated at the event horizon and other universal constant retain their usual designations. Surface gravity for a Schwarzschild BH is given by -

$$K = \frac{c^4}{4GM}$$

Plugging K back in the expression for Hawking Temperature, we get -

$$T_{BH} = \frac{\hbar c^3}{8\pi k_B GM}$$

Notice that the temperature of the blackhole is inversely proportional to its mass. Thus, smaller blackholes are hotter while larger blackholes are cooler. Because we know that $E = Mc^2$, radiating blackholes could equivalently be understood as blackholes losing mass. As the blackhole radiates, it loses some of its mass and gets hotter. Hotter blackholes radiate more (due to to Stefan-Boltzmann Law, as discussed later) and keep shrinking and growing even hotter. This leads to a runaway process called Blackhole evaporation that should emit enormous amounts of energy into the universe as M tends to zero. Such a phenomena has never been observed and the analysis that follows should explain why this has been the case.

Lastly, we now know that radiating blackholes lose mass. This means mass of the blackhole does not remain constant and evolves over time i.e. $M = M(t)$. Therefore,

$$T_{BH}[M(t)] = \frac{\hbar c^3}{8\pi k_B GM(t)} \tag{5}$$

4. **Solving our System**

Finally, I have introduced all the pieces of the problem and can start solving the system. The goal is to model how a blackhole would behave as it radiates hawking radiation and absorbs cosmic microwave background radiation. We would start off by analysing each of the pieces individually and then combining them to get a final time dependent state evolution equation.

4.1 *Setting Up Equations*

Since emission and absorption are both happening via radiation, we use Stefan-Boltzmann Law

$$L = \sigma A T^4$$

But luminosity is nothing but radiant power. So,

$$\frac{dE}{dt} = L = \sigma A T^4$$

From mass-energy equivalence,

$$\frac{dE}{dt} = c^2 \frac{dM}{dt} = \sigma A T^4$$

If we *only* consider the Hawking Radiation and apply the Stefan-Boltzmann Law,

$$\frac{dM_{out}}{dt} = \frac{\sigma}{c^2} * A(M) * T_{BH}(M)^4$$

From earlier results, Eq (4) and Eq (5) give

$$\frac{dM_{out}}{dt} = \frac{\sigma}{c^2} * \left[\frac{16\pi G^2}{c^4} M(t)^2\right] * \left[\frac{\hbar c^3}{8\pi k_B G M(t)}\right]^4$$

On simplifying,

$$\frac{dM_{out}}{dt} = \left[\frac{\sigma \hbar^4 c^6}{256 k_B^4 G^2}\right] \frac{1}{M(t)^2} \tag{6}$$

This is the rate at which the blackhole loses mass due to Hawking Radiation. Now, let us shift our attention towards the mass gained due the absorption of CMBR. In this case, the energy flux is given by

$$F = \sigma T_{CMB}^4$$

But, from the definition of Flux we can get $FA = L = dE/dt = c^2 dM/dt$. So, again we do a similar analysis using the temperature of CMBR.

$$\frac{dM_{in}}{dt} = \frac{\sigma}{c^2} * A(M) * T_{CMB}^4$$

From Eq (4) and Eq (1),

$$\frac{dM_{in}}{dt} = \frac{\sigma}{c^2} * \left[\frac{16\pi G^2}{c^4} M(t)^2\right] * \left[\frac{T_e a(t_e)}{a(t)}\right]^4$$

On simplifying,

$$\frac{dM_{in}}{dt} = \left[\frac{16\pi \sigma G^2 T_e^4 a(t_e)^4}{c^6}\right] \frac{M(t)^2}{a(t)^4} \tag{7}$$

Lastly, it is obvious that

$$\frac{dM_{net}}{dt} = \frac{dM_{in}}{dt} - \frac{dM_{out}}{dt}$$

Plugging in Eq (6) and Eq (7) and defining new constant substitutions will give us the final differential equation that governs how the Blackhole evolves over time

$$\frac{dM_{net}}{dt} = \frac{K_1 M(t)^2}{a(t)^4} - \frac{K_2}{M(t)^2} \tag{8}$$

Here, $K_1 = [16\pi\sigma G^2 T_e^4 a(t_e)^4/c^6]$ and $K_2 = [\sigma\hbar^4 c^6/256 k_B^4 G^2]$.

4.2 *Analytical Results*

In Eq (8), had there not been the $a(t)$ term (or if it was constant), we could have separated the variables on either side of the equality and integrated the ODE (using appropriate substitutions) to get an explicit solution for $M(t)$. This has been attempted in earlier works (Mahulikar S.). However, the introduction of dynamically evolving temperature of CMBR made the differential equation highly non-linear. One analytical treatment that we can do is to set $dM_{net}/dt = 0$ to find the time after which a (non-feeding) blackhole of certain mass reaches equilibrium with CMBR.

We get an analytical result for equilibrium time by assuming a limit solution of the Friedmann equation. Such a limit solution would not be valid for the entire span of the time scale involved. If we were to set up our system in the deep future of our FRW universe, however, then one can reasonably assume that vacuum fraction overpowers all other density terms. In such a case, we would get

$$\frac{dM_{in}}{dt} = \frac{dM_{out}}{dt}$$

On plugging in the derivatives and limit solution of scale factor $a(t) \approx e^{ct\sqrt{\Lambda/3}}$,

$$\frac{K_1 M^2}{e^{t_{eq} 4c\sqrt{\Lambda/3}}} = \frac{K_2}{M^2}$$

On rearranging,

$$\frac{K_1}{K_2} M^4 = e^{t_{eq} 4c\sqrt{\Lambda/3}}$$

Finally, on taking the log of both sides and rearranging further,

$$t_{eq} = \frac{1}{4c\sqrt{\Lambda/3}} \ln\left(\frac{K_1 M^4}{K_2}\right)$$

4.3 *Computational Results*

Finally, I decided to numerically integrate the final state evolution differential equation to model our Blackhole in `python`. I used the `astropy` library to check the dimensional validity of the equations that I had worked out. The dimensions of the LHS of Eq (8) were $[M][T]^{-1}$. By using the `astropy.units` and `astropy.constants` package, I verified the following -

1. The units of $K_1 * M^2$ were Ws^2/m^2, which is equivalent to Kg/s. i.e. $[M][T]^{-1}$
2. Similarly, the units of K_2/M^2 were Kg/s i.e. $[M][T]^{-1}$
3. The scale factor $a(t)$ was verified to be dimensionless.

I used the RK4 numerical integration scheme to simulate the system. As mentioned earlier, the intial conditions were $t_0 = 0$, $a(t_0) = 1$, $\dot{a}(t_0) = H_0$, $M(t_0) = 10M_\odot$ and $T(t_0) = 2.75K$. From t_0, I ran my integrator backwards until $a(t) < 0.0009$ and forward for 40 billion years (or 40 Giga-annum) in the future. Now, we have everything we need to run the simulation.

5. **Conclusion**

The result of numerically integrating Friedmann Equation is displayed in Fig. 3. At $t = 0$ (current time), the scale factor is 1 and as the integration was run backwards in time, scale factor started asymptotically approaching zero rapidly about 13.5 billion years ago. This is also roughly what cosmologists suspect might be the age of our universe.

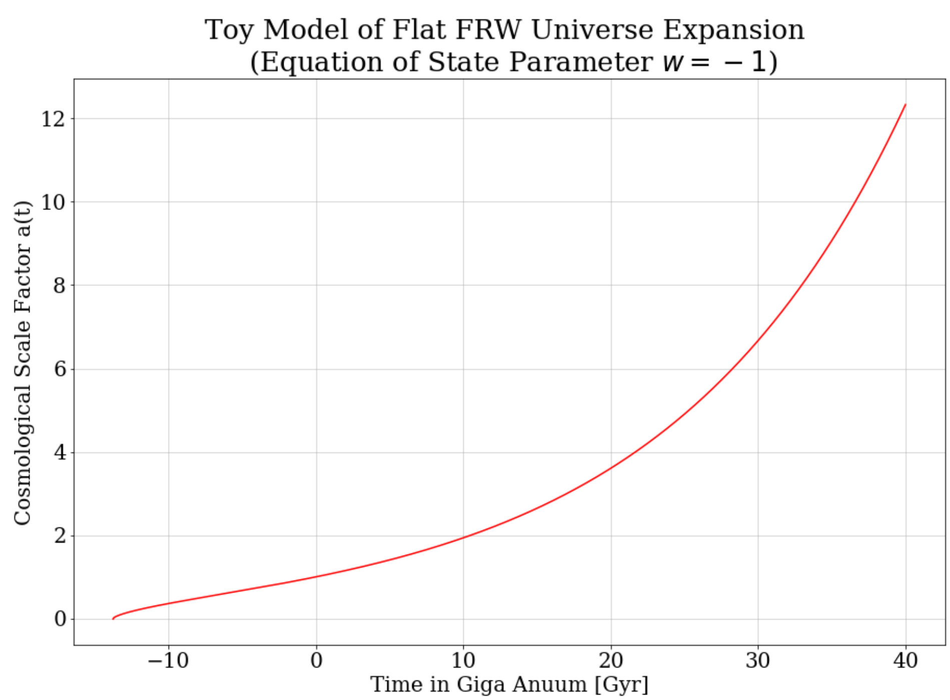

Figure 3: Evolution of Scale Factor of the universe when run backwards approaches a singularity roughly 13.5 billion years ago

Two clear inflection points are noticeable in the curve when analysed carefully and these represent transitions between radiation dominant era to matter dominant era and, then, from matter era to dark-energy dominant era. 40 billion years in the future, my simulated universe seems to be 12 times its current size.

With a model for the evolution of scale factor, the next thing I tested was the temperature of CMBR. The results are displayed in Fig. 4. The change in temperature spanned orders of magnitude and to properly express the change I decided to plot a semi-log graph. In accordance with the popular understanding, there was a rapid drop in temperature a few years after the Big Bang. The temperature drop has slowed down now and the temperature is asymptotically approaching zero in the far future.

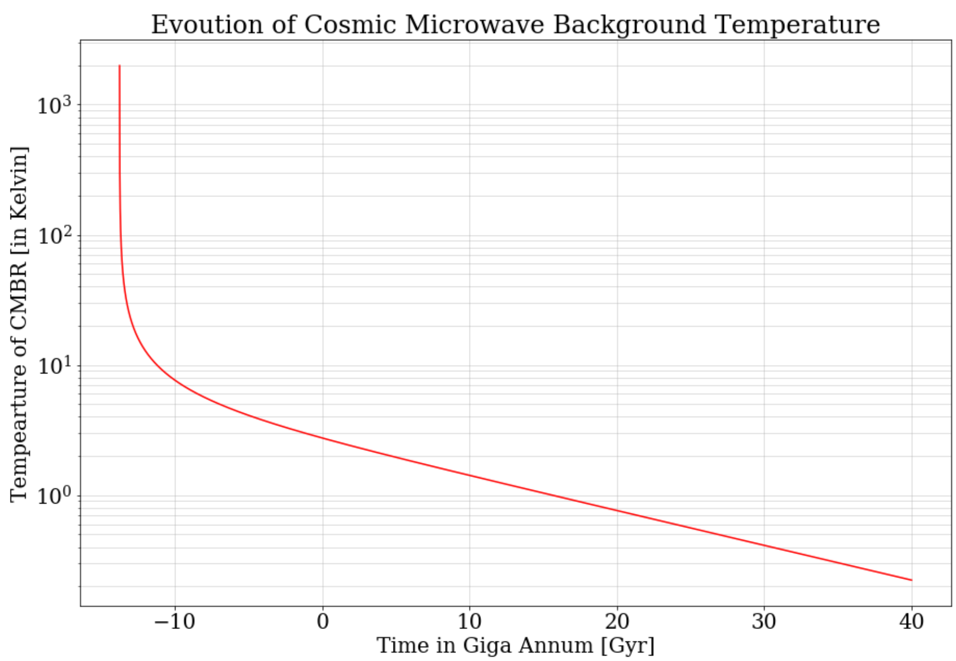

Figure 4: Evolution of Scale Factor of the universe when run backwards approaches a singularity roughly 13.5 billion years ago

CMB corresponds to a redshift factor of $z \approx 1100$ (Carroll and Ostlie). I manually plugged in the redshift factor and back-calculated using the program to find the temperature of CMBR at $z = 1100$ to be roughly 2000 K. This is off from what Saha Equation predicts (3000K) but this loss in accuracy is to be expected, partly due to error in numerical integration and partly (rather, primarily) due to the system's inefficiency to model the radiation dominant era well. This is why, I took extra care that I place my blackhole evolution analysis away (in time) from radiation dominant era.

Finally, I modelled the complete mass evolution equation. This was, ironically, the most underwhelming result (in some sense). My program kept plotting a straight line representing constant mass through time. When I calculated manually using the program, the change in mass over a period of 50 billion years remained zero.

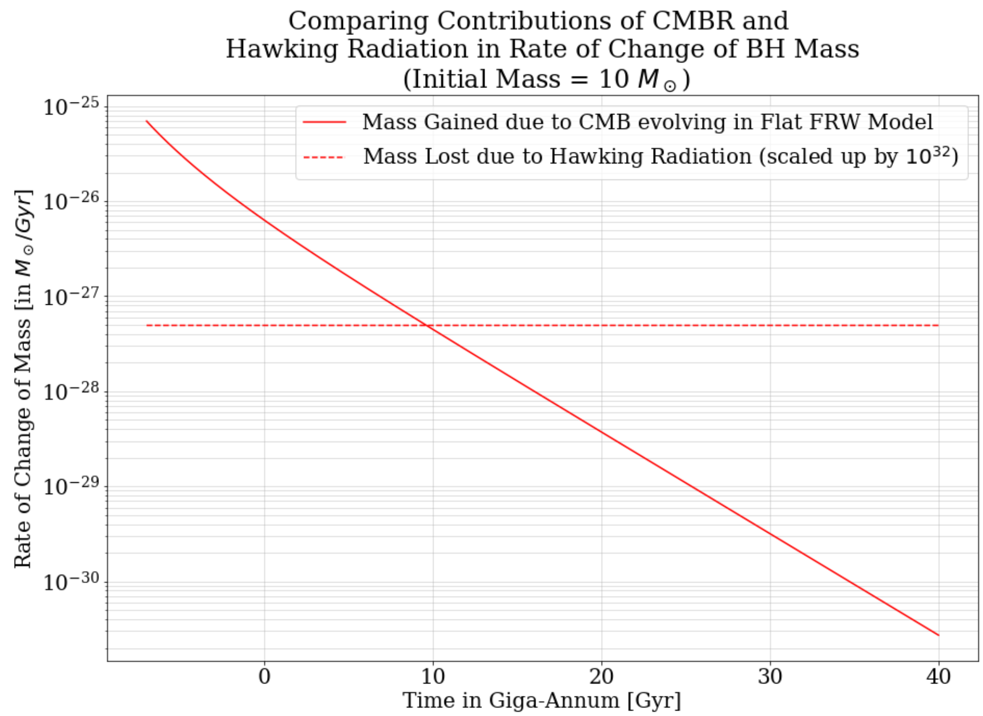

Figure 5: Mass Gain and Mass Loss rates of a Black-Hole of $10M_\odot$ (as of 7 Billion Years in the past from now) studied for 47 Billion Years

After checking my theory and code multiple times, I can confidently assert the final findings of my study. To understand the time evolution of BH, it is more instructive to visualize the individual mass change rates across time as done in Fig. 5. The first thing that is clearly evident is that the mass lost due to Hawking radiation is an extremely small quantity of the order of $10^{-60}M_\odot/Gyr$ (or 10^{-39} Kg/yr). This was, at least to me, a surprising result. A non-feeding blackhole which is 10 times the mass of our sun, loses less mass in a year due to Hawking Radiation than the mass of an electron. To even have the $\dot{M_{out}}$ quantity show up on the graph, I had to scale it up by a factor of 10^{32}. Why it appears to be constant would also be clear in a moment.

Turning our attention to the mass gain rate, we see it spans over multiple orders of magnitude as the scale factor evolves and is much greater than the mass loss rate. However, even the mass gain rate is of roughly of the order of 10^{-25} to $10^{-35}M_\odot/Gyr$. Compared to the mass of the blackhole ($10M_\odot/Gyr$), these quantities are extremely minuscule. The reason why the mass of the blackhole appeared constant over almost 50 billion years was because the changes were happening after the 25th decimal place and python was clearly not working under that level of precision. Since the

Hawking radiation depends on mass of the blackhole and the mass remained constant throughout the simulation, the intensity of Hawking radiation remained constant too.

I tried to nondimensionalize the ODE in hopes of finding a scale factor that helps in demonstrating some behavior of the system. However, modelling even the nondimensional ODE still gave constant mass results as before. The behavior of the mass of a non feeding blackhole in a CMBR bath on the scale of Giga years is simple - *almost* nothing happens to it.

However, it is immediately clear that even an orphan blackhole in the middle of nowhere that does not have any matter to feed on, would gain mass much more quickly than the rate at which it will lose mass. This explains why we have never observed Black hole evaporation until now; such events require initial conditions that are extremely unlikely to occur. Further, the results make it clear that blackholes are some of the most robust astrophysical objects that function on timescales much larger than the age of the universe itself.

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