Learner's Map of Numerical Relativity I: Canonical General Relativity

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Pre-requisites: Undergraduate-level familiarity with concepts in GR like curvature, Lie derivatives, field equations and their solutions.

I. Introduction

With the recent cabinet approval for LIGO India's funding, it is an exciting time for GR community. All the more relevant, then, is to understand that a significant reason why LIGO detectors manage to consistently locate needles in (astronomically large) haystacks is because they have a sense of what they are looking for. Linearized perturbations on metric field are insufficient to make precise predictions about gravitational waves due to mergers. Therefore, *Numerical Relativity* plays an important role in modelling wave-forms and this is what I discuss in this short note.

A pragmatic impetus for studying numerical schemes for understanding gravity is that the mathematics involved in GR is very difficult and linearization/perturbative techniques fail to capture the subtle features of certain astrophysical phenomena like gravitational waves. A philosophical reason is that GR presents a block universe in the form of a 4 dimensional spacetime manifold which exists as an eternal object with no dynamics. Since humans are constrained on the phenomenological slices of the present, we require a general method of mapping the theoretical features to domains we perceive and comprehend. Numerical relativity (NR), beyond a working knowledge of computational methods, also requires a strong grip on certain theoretical concepts like canonical GR and conformal decompositions which are often not covered in undergraduate (or even graduate courses) in GR. In this document, I present an overview of these theoretical concepts which are also often useful in domains unrelated to NR (such as quantization of gravity or cosmology).

II. Setting the Stage

To set the stage, we observe that the twice contracted Bianchi identity gives us the following property of the Einstein tensor

$$\nabla_\mu G^{\mu\nu} = 0 \tag{1}$$

$$\implies \partial_0 G^{0\nu} + \partial_k G^{k\nu} + G^{\lambda\nu}\Gamma^{\mu}_{\lambda\mu} + G^{\mu\lambda}\Gamma^{\nu}_{\lambda\mu} = 0$$

Notice that the Einstein tensor contains at most second time derivatives and the second term in the above expansion is a spatial derivative of the Einstein tensor. Therefore, all terms in the expansion must contain at most second time derivatives. However, the first term is already a time derivative. This means the $G^{0\nu}$ and $G^{\mu 0}$ components of the Einstein tensor contains only first order time derivatives. This observation helps in categorizing the ten independent field equations of general relativity in two classes -

1) Four under-determined elliptic type PDEs which serve as constraint equations (first time derivatives)
2) Six under-determined hyperbolic type PDEs which serve as evolution equations (second time derivatives)

This division into constraint and evolution equations is often explained by referring to the similar division in Maxwell's equations where the divergence equations constraint the set of possible solutions (or allowed initial conditions) and the curl equations give time-evolution of initial data. The under-determined aspect of the equations indicates that there exists a gauge freedom[1] in the theory. Solving the field equations is essentially solving an Initial-Boundary value problem. In the following sections, we sharpen this observation by decomposing objects in 3+1 dimensions.

III. Canonical General Relativity

A. Decomposing Metric in 3+1 Dimensions

Via appropriate coordinate choices, we can represent the 4D spacetime as a stack of 3-dimensional hypersurfaces Σ (parameterized some t) foliating it[2]. The appropriate coordinate choice results in a metric of the form

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

The metric components carry a very geometric interpretation (see Fig. 1).

1) Lapse Function α: Helps in connecting the time elapsed in coordinate frame dt with the time lapsed $d\tau$ for

[1] More formally, one can say the solution cannot be determined to more than a diffeomorphism

[2] Not all torsion free curved manifolds can be completely foliated with hypersurfaces but sufficiently many do to remain interesting.

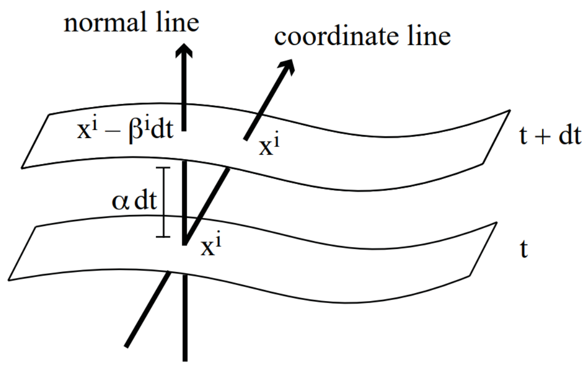

Fig. 1. Geometrical interpretation of the various components making up the 3+1 decomposed metric visualized on two adjacent hypersurfaces. Figure taken from [1]

an observer whose worldline is perpendicular to the hypersurface (called Eulerian or Normal observers).

$$d\tau = \alpha(t, x^i)dt \tag{2}$$

2) Shift Vector β_i: Captures the relative velocity with which the coordinate lines on the hypersurface drift away from the Eulerian observer.

$$x^i_{t+dt} = x^i_t - \beta^i(t, x^i)dt \tag{3}$$

3) Spatial Metric $\gamma_{ij}(t, x^i)$: Defines the invariant line element on the spatial hypersurface Σ

$$dl^2 = \gamma_{ij}dx^i dx^j \tag{4}$$

There are infinite ways to foliate the spacetime. The freedom in the foliation via stacking hypersurfaces as well as the coordinatizing of the hypersurface corresponds to the previously mentioned gauge freedom in the theory. Therefore, lapse function α and shift vector β^i are known as the gauge functions. The spacetime interval in terms of the decomposed metric then takes the following form

$$\boxed{ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j} \tag{5}$$

B. Decomposing Curvature in 3+1 Dimensions

Now that we have our 3+1 decomposed metric, let us try to decompose our curvature components. GR is mostly concerned with the intrinsic curvature of manifolds (constructed using second derivatives of the metric) since first derivatives can be made to vanish locally. However, in our decomposition, the embedding of the hypersurfaces and, consequently, the extrinsic curvature of the metric become important.

1) Extrinsic Curvature: The extrinsic curvature of the hypersurface K_{ij} is a measure of the change in the normal vector under parallel transport.

$$K_{ij} = \frac{1}{2\alpha}(\nabla_i\beta_j + \nabla_j\beta_i - \partial_t\gamma_{ij}) \tag{6}$$

On identifying the lie derivative $\mathcal{L}_\beta\gamma_{ij} = \nabla_i\beta_j + \nabla_j\beta_i$ and rearranging, we arrive at the kinematic evolution equation.

$$\boxed{(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}} \tag{7}$$

A discussion on each of the boxed equations follows in Section IV. For now, let us continue assembling the remaining pieces.

2) Intrinsic Curvature: Understanding the projection of the 4 dimensional Riemann tensor to 3+1 dimensions is not essential at this point but one must know that the 4 dimensional intrinsic curvature can be written in terms of the lapse, shift and extrinsic curvature using the Gauss-Codazzi-Mainardi set of equations.

$$R^{(4)}_{\mu\nu\rho\sigma} \leftrightarrow R^{(3)}_{\mu\nu\rho\sigma}$$

C. Projections of Einstein Field Equations

The field equations of general relativity are

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{8}$$

Though the following results can be generalized for matter solutions, for the sake of simplicity we will only look at the vacuum solutions i.e. $T^{\mu\nu} = 0$ for now (which still provide many interest investigations).

When we talk of projecting the field equations, for each index we have two choices. We can project components onto the hypersurface by contracting with a projection operator defined as

$$P^\mu_\nu \equiv \delta^\mu_\nu + n^\mu n_\nu \tag{9}$$

Or, we can project in direction perpendicular to the hypersurface by contracting the Einstein tensor's index with the normal vector n^μ.

$$n^\mu = (1/\alpha, -\beta^i/\alpha) \text{ and } n_\mu = (-\alpha, 0) \tag{10}$$

By making different choices for field equation projections, we arrive at various important results. Further, unlike deriving the Gauss-Codazzi relations, performing the following calculations is fairly simple and instructive.

D. Hamiltonian Constraint

If we contract both the indices using the normal vector, we get

$$\boxed{2G_{\mu\nu}n^\mu n^\nu = R^{(3)} - K_{ij}K^{ij} + K^2 = 0 \equiv \mathcal{H}} \tag{11}$$

where K is the contraction of the extrinsic curvature $K = \gamma^{ij}K_{ij}$. This equation is known as the Hamiltonian constraint.

E. Momentum Constraint

On projecting one index onto the hypersurface and the other on the normal, we get

$$\boxed{-P^\mu_i n^\nu G_{\mu\nu} = \nabla^j K_{ij} - \nabla_i K \equiv \mathcal{M}_i} \tag{12}$$

where $\nabla_i \equiv (P\nabla)^\mu_i$ is the projection of the covariant derivative. These three equations together are known as the momentum constraints.

F. Dynamical Evolution Equation

Finally, if we project both of the indices onto the hypersurface, we get

$$P_i^\mu P_j^\nu G_{\mu\nu} = \mathcal{L}_n K_{ij} \tag{13}$$

$$\boxed{(\partial_t - \mathcal{L}_\beta)K_{ij} = -\nabla_i\nabla_j\alpha + \alpha\left(R_{ij}^{(3)} + K_{ij}K - 2K_i^k K_k^j\right)}$$

which is known as the dynamical evolution equation.

IV. Checkpoint: ADM-York Equations

Through the decomposition of the metric and projecting the field equations, we have assembled all the pieces of ADM formalism which is a canonical formulation of GR. Let us briefly recapitulate what we have gathered till now.

The three momentum constraints Eq. (12) and one Hamiltonian constraint Eq. (11) together constitute the four elliptic type PDEs which must be solved to get the initial data. Notice there are no time derivatives in these equations. Further, Bianchi identities ensure (at least in theory) that if the initial data meets the constraint requirements, time evolution would not break that fulfillment.

Once we have the initial data, we can use the evolution equations Eq. (7) and Eq. (13) to evolve the solution towards the next spatial hypersurface.

Hamiltonian Formulation [2]:

Canonical General Relativity is often introduced as the hamilitonian form of general relativity. However, in our current set-up, it is not immediately transparent why the ADM equations are the hamilitoninan equations. This is because the presentation was not of the original ADM equations (which were written in terms of conjugate momenta π) but instead York's form of the ADM equations (also called ADM-York equations). However, to get a quick intuition, one can think of the following. Hamiltonian mechanics involves two first order time derivative equations for the phase space coordinates (q, p). The evolution equations which we have presented are both first order in time. However, the extrinsic curvature K_{ij} already has first order derivatives of the spatial metric γ_{ij} in it. Therefore, if we treat the metric components as an analogue of q, then the extrinsic curvature can be thought of as an analogue of the conjugate momenta p. Thus, the kinematic evolution equation Eq. (7) is the first type of the Hamiltonian equations of motion by being the first-order time derivatives of the metric. The dynamical evolution equation is the second type of the hamiltonian equations of motion by being the first-order time derivative of the first-order derivatives of the metric.

V. Well-Posedness

Though we have successfully separated our constraint equations from the evolution equations, we are not yet in a position to effectively perform numerical simulations. A major issue which has been identified is that the ADM-York equations are not well-posed [3]. This means even if we start with a small perturbation, the equations eventually blow up. A well-posed system (like the wave equation) shows 'global hyperbolicity'[3] i.e.

$$|u(t, x)| \leq ke^{\alpha t}|u(0, x)| \tag{14}$$

for $\partial_t u = \mathcal{D}u$ where $\mathcal{D}$ is some differential operator.

For any numerical simulation, we will require a well-posed form of the field equations and it is often said in NR 'more formalisms have been proposed which claim global hyperbolicity than there are groups to test them'. One way to arrive at well-posedness is by selecting the appropriate gauge functions (such as Harmonic Gauge, Bonna Masso, etc.) [4]. However, an important and instructive well-posed system of NR equations is the BSSNOK (or sometimes just BSSN) formulation. BSSN formalism is based on the conformal recasting of the ADM elements (which is discussed in the part two of my Learner's Map to NR series).

VI. Discussion

The next part of the Learner's Map covers how various NR elements (metric, curvature, field equation, ADM equations, etc.) are conformally decomposed to achieve well-posedness. This also naturally leads to a discussion of how Black-Hole initial data is solved for. This would complete an elementary bird's eye view of the theoretical aspects of numerical relativity. Subsequent parts of this *Learner's Map*, once published, can be accessed online - kartiktiwari.in/nr.

In this brief note, the numerical schemes required to solve the equations were not discussed. Writing high resolution parallel scripts is an art and science in itself. However, readers who do not yet wish to spend weeks writing simulations can opt to explore existing open-source NR libraries. There are many but one which stands out for its user-friendliness and versatility is EinsteinToolkit (ETK). ETK is based on the Cactus environment and each module is called a thorn. It also employs CarpetX for adaptive-mesh refinement. An effective entry point is to try running example simulations and slowly learning to tweak parameters. One such example simulation which can be run easily (if one has appropriate cluster resources) is the GW150914 event (see Fig. 2). ETK gallery expands with every major release and, as a part of the NRCSS hackathon, my team and I also ran a low resolution simulation of a Fishbone-Moncrief disk (initial conditions developed by Zach Eitnee). FM disks are toroidal self gravitating viscous matter around Kerr blackholes (see Fig. 3).

Beyond the cited references, those interested are encouraged to watch the lectures on Numerical Relativity delivered by Prof. Thomas Baumgarte at ICTS, Prof. Helvi Witek at IPAM and the Canonical GR lectures during the Heraeus Winter School on Gravity and Light, all of which can be easily found on the Youtube.

Acknowledgment

I am grateful to Prof. Miguel Alcubierre for his guidance in my NR engagements during the summer of 2021, Prof.

[3]For those interested, a rigorous analysis of hyperbolicity can be performed using eigenfields.

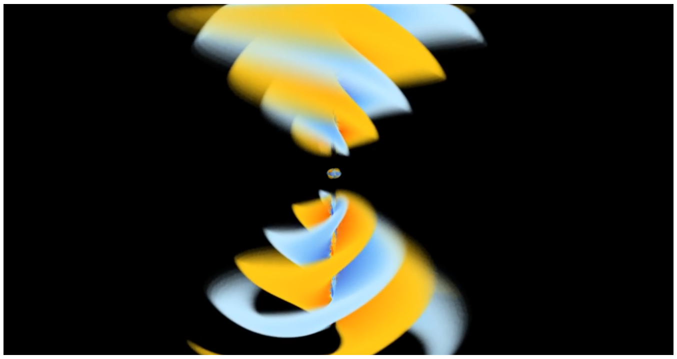

Fig. 2. Screenshot from the example simulation of binary black hole merger run on EinsteinToolkit

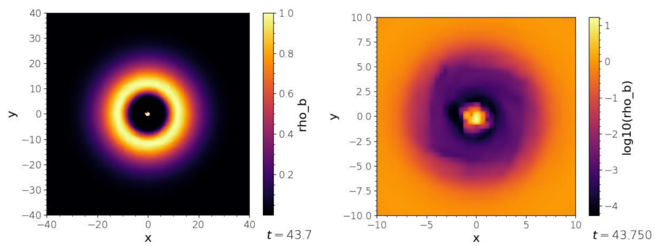

Fig. 3. Low resolution simulation of Fishbone-Moncrief disk done for inclusion on the ETK Gallery page. Visualization of the cross section of the toroid.

Dipankar Bhattacharya for teaching me how to think like a computational physicist and Prof. Vikram Vyas and Prof. Suratna Das for my undergraduate courses in General Relativity.

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